ECE 313 Lecture Notes

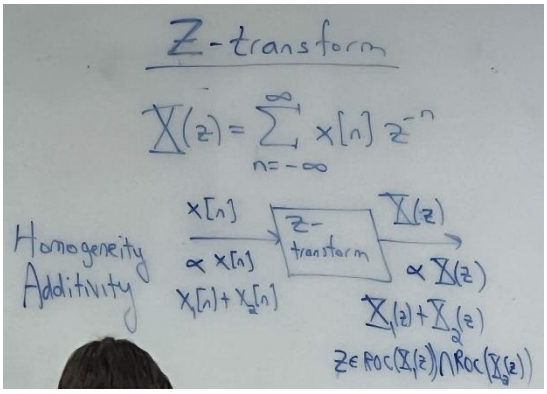
Lecture 10 on Z-transforms, Part 2

Thursday, October 29, 2026

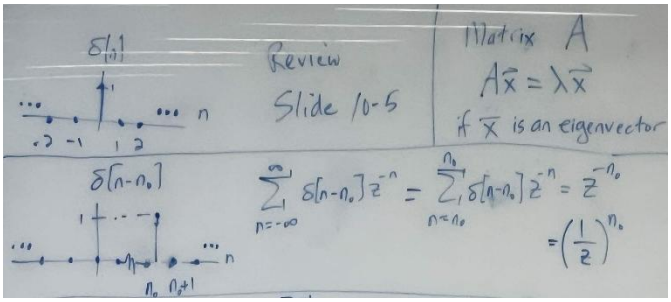
By TA Alice Liu

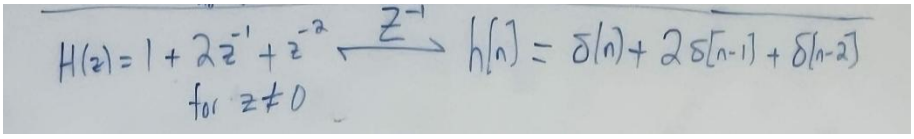
…

Z-Transform



- When viewed as a system that takes input $x[n]$ and produces output $X(z)$, is the z-transform Linear? YES
- A linear system must satisfy both
  - Homogeneity
  - Additivity
- When computing the z-transform, we must track the "Region of Convergence", which is the set of complex z values for which the z-transform converges
- Z-Transform Pairs
  - Discrete-time impulse $\delta[n]$
  - Time Shift $\delta[n - n_0]$
- Inverse z-transform:
  - Important Note: $z \neq 0$ -> specifies "Region of Convergence"





…

Converting from the Z Domain to Discrete Time Frequency Domain

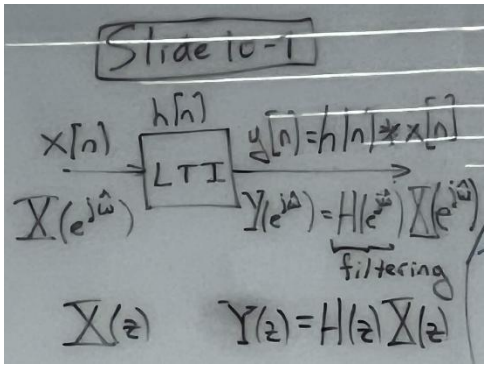
- Z Domain: a generalized frequency domain
- Whenever the region of convergence includes the unit circle, we can substitute $z = e^{j\,\hat{\omega}}$ into the z-transform $H(z)$ to obtain the frequency response (more below)

…

Z-Transform Properties

- We must track the Region of Convergence (ROC) for all properties!
- Superposition
- Time-Delay

- Convolution and the Z-Transform (Convolution Property)
    - Convolution in the TIME domain ↔ Multiplication in the Z-domain
        - This is the basis of filtering!
            - For manual design of filters, we tend to be in the *z*-domain
            - For automated design of filters, we tend to be in the Frequency or Time domain, and some are in both domains
- Q&A: We have the original summation; on the inside we plug in the formula for y[n]. Why?
    - First equation: y[n], the convolution of impulse response h[n] and input x[n]
    - Second equation: frequency response of y[n] is the product of the frequency responses of h[n] and x[n]
    - Third equation: z-transform of y[n] is the product of the z-transforms of h[n] and x[n]
    - We go to the frequency domain to make it easier to analyze the effect of the filter on an input signal and the *z* -domain to make it easier to design filters



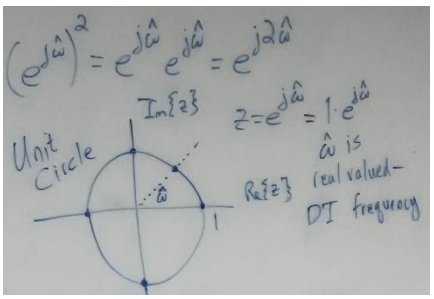
...

Cascading LTI Systems

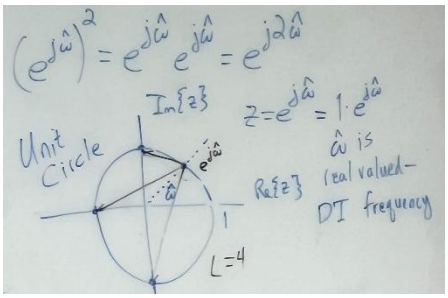
- Allows flexibility of designing each individual LTI system
- But! We need to ensure...
    - LTI in **hardware** implementation as well
    - Initial Conditions being satisfied
    - We must allocate enough bits for every multiplication, addition, and other operation to retain all the bits produced by the calculation in the worst case
    - Any truncation or rounding of results causes a loss of bits (nonlinearity) and we lose LTI properties
    - For an FIR filter, we can allocate a finite number of bits to hold all data and computations without losing any bits
- Cascaded LTI Systems can be re-written/re-represented as ONE LTI System without changing the output
- The order of the LTI Systems in the cascade can be switched

...

Z-Domain to Frequency Domain (a.k.a. “Fourier” Domain)



- In polar form: $z = r\, e^{j\omega}$
  - r is the magnitude (or radius) of the vector on the Unit Circle
  - ω is the phase of the vector on the Unit Circle
- Rewrite into magnitude/phase format to better connect to the Unit Circle ($r = 1$)
- Alternatively, since the Unit Circle can be interpreted in Cartesian coordinates as having real (x-axis) and imaginary (y-axis) components…
  - we can plot z on the Unit Circle by re-writing it as a complex number:
    - $z = a + jb$
    - where $Real\{z\} = a$ and $Imag\{z\} = b$
- Each root of the FIR filter’s transfer function is on the unit circle
  - A “zero” on the unit circle eliminates the frequency in the input signal at the angle of the zero and nearby frequencies are attenuated
  - For an *L*-point averaging filter,



    - we’ll have = $L - 1$ zeros
    - lowpass filter
    - first null-frequency $2\pi/L$
    - “null” bandwidth $\frac{2\pi}{L}$
    - zeros or “nulls” appear at integer multiples of the $\frac{2\pi}{L}$
    - When *L* is odd, half of zeroed-out frequencies are positive and the other half are negative.
    - When *L* is even, the discrete-time frequency at π is zeroed out; of the remaining *L-2* zeroed-out frequencies, half are positive and half are negative.

FIR Nulling Filter (after lecture)

- To remove a specific positive discrete-time frequency w_0, we would need a three-point FIR filter where the roots of the transfer function for the FIR filter are at
- $z_0 = \exp(j\, w_0)$ and $z_1 = \exp(-j\, w_0)$
- The transfer function becomes
  $H(z) = (1 - z_0\, z^{-1})\,(1 - z_1\, z^{-1}) = 1 - (z_0 + z_1)\, z^{-1} + z^{-2}$
- By replacing the formulas for z_0 and z_1
  $H(z) = 1 - 2\cos(w_0)\, z^{-1} + z^{-2}$
- This means that FIR filter coefficients are
  `1, -2 cos(w0), 1`